Law of large number and central limit theorem under uncertainty, the related new Ito’s calculus and applications to risk measures

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Let \( S_n = \sum_{i=1}^{n} X_i \) where \( \{X_i\}_{i=1}^{\infty} \) is a sequence of independent and identically distributed (i.i.d.) of random variables with \( E[X_1] = \mu \). According to the classical law of large number (LLN), the sum \( S_n/n \) converges strongly to \( \mu \). Moreover, the well-known central limit theorem (CLT) tells us that, with \( \mu = 0 \) and \( \sigma^2 = E[X_1^2] \), for each bounded and continuous function \( \varphi \) we have

\[
\lim_{n \to \infty} E[\varphi(S_n/\sqrt{n})] = E[\varphi(X)] \text{ with } X \sim N(0, \sigma^2).
\]

These two fundamentally important results are widely used in probability, statistics, data analysis as well as in many practical situation such as financial pricing and risk controls. They provide a strong argument to explain why in practice normal distributions are so widely used. But a serious problem is that the i.i.d. condition is very difficult to be satisfied in practice for the most real-time processes for which the classical trials and samplings becomes impossible and the uncertainty of probabilities and/or distributions cannot be neglected.

In this talk we present a systematical generalization of the above LLN and CLT. Instead of fixing a probability measure \( P \), we only assume that there exists a uncertain subset of probability measures \( \{P_\theta : \theta \in \Theta\} \). In this case a robust way to calculate the expectation of a financial loss \( X \) is its upper expectation: \( \mathbb{E}[X] = \sup_{\theta \in \Theta} E_\theta[X] \) where \( E_\theta \) is the expectation under the probability \( P_\theta \). The corresponding distribution uncertainty of \( X \) is given by \( F_\theta(x) = P_\theta(X \leq x), \theta \in \Theta \). Our main assumptions are:

1. The distributions of \( X_i \) are within an abstract subset of distributions \( \{F_\theta(x) : \theta \in \Theta\} \), called the distribution uncertainty of \( X_i \), with \( \overline{\mu} = \mathbb{E}[X_i] = \sup_{\theta} \int_{-\infty}^{\infty} x F_\theta(dx) \) and \( \underline{\mu} = -\overline{\mathbb{E}}[-X_i] = \inf_{\theta} \int_{-\infty}^{\infty} x F_\theta(dx) \).
2. Any realization of \(X_1, \ldots, X_n\) does not change the distributional uncertainty of \(X_{n+1}\) (a new type of ‘independence’).

Our new LLN is: for each linear growth continuous function \(\varphi\) we have
\[
\lim_{n \to \infty} \hat{E}[\varphi(S_n/n)] = \sup_{\mu \leq v \leq \mu} \varphi(v).
\]
Namely, the distribution uncertainty of \(S_n/n\) is, approximately, \(\{\delta_v : \mu \leq v \leq \mu\}\).

In particular, if \(\mu = \mu = 0\), then \(S_n/n\) converges strongly to 0. In this case, if we assume furthermore that \(\sigma^2 = \hat{E}[X_i^2]\) and \(\sigma^2 = -\hat{E}[-X_i^2]\), \(i = 1, 2, \ldots\). Then we have the following generalization of the CLT:
\[
\lim_{n \to \infty} [\varphi(S_n/\sqrt{n})] = \hat{E}[\varphi(X)], \quad L(X) \in N(0, [\sigma^2, \sigma^2]).
\]
Here \(N(0, [\sigma^2, \sigma^2])\) stands for a distribution uncertainty subset and \(\hat{E}[\varphi(X)]\) its the corresponding upper expectation. The number \(\hat{E}[\varphi(X)]\) can be calculated by defining \(u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]\) which solves the following PDE \(\partial_t u = G(u_{xx})\), with \(G(a) := \frac{1}{2}(\sigma^2 a + \sigma^2 a^-)\).

An interesting situation is when \(\varphi\) is a convex function, \(\hat{E}[\varphi(X)] = E[\varphi(X_0)]\) with \(X_0 \sim N(0, \sigma^2)\). But if \(\varphi\) is a concave function, then the above \(\sigma^2\) has to be replaced by \(\sigma^2\). This coincidence can be used to explain a well-known puzzle: many practitioners, particularly in finance, use normal distributions with ‘dirty’ data, and often with successes. In fact, this is also a high risky operation if the reasoning is not fully understood. If \(\sigma = \sigma = \sigma\), then \(N(0, [\sigma^2, \sigma^2]) = N(0, \sigma^2)\) which is a classical normal distribution. The method of the proof is very different from the classical one and a very deep regularity estimate of fully nonlinear PDE plays a crucial role.

A type of combination of LLN and CLT which converges in law to a more general \(N([\mu, \mu], [\sigma^2, \sigma^2])\)-distributions have been obtained. We also present our systematical research on the continuous-time counterpart of the above ‘\(G\)-normal distribution’, called \(G\)-Brownian motion and the corresponding stochastic calculus of Itô’s type as well as its applications.