Categorical crepant resolutions of higher dimensional simple singularities

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  (algebraic and geometric structure on a set)
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  triangulated category: shift functor $[1]$, distinguished triangles $a \to b \to c \to a[1]$ instead of exact sequences $0 \to a \to b \to c \to 0$. 
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- Coh(X) is a heart of $D^b(\text{Coh}(X))$ w.r.t. $t$-structure. There are many hearts in a triangulated category.
- Example: $D^b(\text{Coh}(\mathbb{P}^n)) \cong D^b(\text{Mod-}R)$ for a finite dimensional non-commutative $k$-algebra $R$ (representations of a quiver algebra).
1. finite type: $\dim_k \sum_{p \in \mathbb{Z}} \text{Hom}^p(a, b) < \infty$.

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2. **saturated**: $\forall$ exact functor $F : D^b(\text{Coh}(X)) \rightarrow D^b(\text{Coh}(P))$, $\exists a \in D^b(\text{Coh}(X))$ s.t. $F(b) \cong \text{Hom}(a, b)$ (representable)
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3. **Serre functor**: $S \in \text{Aut}(D^b(\text{Coh}(X)))$,
   $\text{Hom}(a, b) \cong \text{Hom}(b, S(a))^*$.
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Yusuke Okamura  University of Tokyo  Categorical crepant resolutions of higher dimensional simple sing
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Example: If \( X \) is \( n \)-dim Calabi-Yau variety, \( S \cong [n] \) \((n \text{-Calabi-Yau category})\)
Minimal models of surfaces

- $X$: algebraic surface; smooth projective variety of dimension 2.
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- *minimal model*: no more $(-1)$-curve
Minimal models of surfaces and exceptional objects

- $c_0 = \mathcal{O}_C(-1) \in D^b(\text{Coh}(X))$: $\sum_{p \in \mathbb{Z}} \text{Hom}^p(c_0, c_0) \cong k$ as graded rings (exceptional object)
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- $f_*c_0 \cong 0$.
- $D^b(\text{Coh}(X')) \cong \{ b \in D^b(\text{Coh}(X)) | \text{Hom}^p(b, c_0) = 0, \forall p \}$ (left orthogonal complement of $\langle c_0 \rangle$: $b \perp c_0[p]$)
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- $\forall a \in D^b(\text{Coh}(X))$, $b \rightarrow a \rightarrow c \rightarrow b[1]$. $c \in \langle c_0 \rangle$, $b = f^* f_* a \in D^b(\text{Coh}(X'))$. 

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- Remark: \( D^b(\text{Coh}(X)) \) has no *orthogonal* decomposition.
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- Remark: \( D^b(\text{Coh}(X)) \) has no orthogonal decomposition.
- Corollary: If \( n \)-Calabi-Yau category, no SOD.
- Proof: If \( \mathcal{B} \perp \mathcal{C} \), then \( \mathcal{C} \perp \mathcal{B} \).
Minimal model program

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X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m
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\[f_i : X_{i-1} \rightarrow X_i\] birational map, one of the following:

1. (D): contraction of codimension 1 subvariety (\textit{divisorial contraction})

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- Canonical divisor decreases in both cases: \( \mu_{i-1}^* K_{X_{i-1}} > \mu_i^* K_{X_i} \)
on a common resolution.
One of the following output:

1. (MM): $K_{X_m}$ is nef, $(K_{X_m} \cdot C) \geq 0$, $\forall C$. \textit{(minimal model)}

2. (MF): $f : X_m \to Y$, $(K_{X_m} \cdot C) < 0$, $\dim Y < \dim X_m$, $\forall C$ in a fiber of $f$. \textit{(Mori fiber space)}
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- *relative version* of MMP over $S$: starting from $h : X \to S$, all maps are over $S$.

- Example: If $h : X \to S$ arbitrary resolution of singularities, a relative minimal model $h_m : X_m \to S$ is a *minimal resolution*. 
MMP and semi-orthogonal decomposition (Example 1)

- $f : X \to X'$ blowing-up of a smooth variety along a smooth subvariety $E' \subset X'$. (typical example of a divisorial contraction).
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- Canonical divisors: $K_X - f^*K_{X'} = nE$.
- Corresponding SOD: For $f_E = f|_E$, $i : E \to X$, $D^b(\text{Coh}(X)) = \langle i_*(f_E^*D^b(\text{Coh}(E'))) \otimes O_E(-n), \ldots, i_*(f_E^*D^b(\text{Coh}(E'))) \otimes O_E(-1), f^*D^b(\text{Coh}(X')) \rangle$. [Bondal-Orlov]
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- Corresponding SOD: $D^b(\text{Coh}(X)) = \langle i_*\mathcal{O}_{\mu(E)}(-n + n'), \ldots, i_*\mathcal{O}_{\mu(E)}(-1), \mu_*(\mu')^*D^b(\text{Coh}(X')) \rangle$.
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- Corresponding SOD: $D^b(\text{Coh}(X)) = \langle i_* \mathcal{O}_\mu(E)(-n + n'), \ldots, i_* \mathcal{O}_\mu(E)(-1), \mu_*(\mu')^* D^b(\text{Coh}(X')) \rangle$.
  [Bondal-Orlov]

- If $n = n'$, $f : X \dashrightarrow X'$ is a flop.

- $\mu_*(\mu')^* : D^b(\text{Coh}(X)) \cong D^b(\text{Coh}(X'))$. 
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In particular, if $\mu^* K_X = (\mu')^* K_{X'}$, then $D^b(\text{Coh}(X)) \cong D^b(\text{Coh}(X'))$. 
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\[ \pi : X' \to X' : \text{associated DM stack, } \tilde{\mathcal{X}} = X \times_{X'} X', \mu : \tilde{\mathcal{X}} \to X, \mu' : \tilde{\mathcal{X}} \to X'. \]

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Yuujiro Kawamata  University of Tokyo  Categorical crepant resolutions of higher dimensional simple sing...
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- *Terminal* singularities: for every resolution $f : Y \to X$, $K_Y - f^* K_X \geq 0$ and contains all exceptional divisors ($K_Y - f^* K_X = \sum e_j E_j$, $e_j > 0$, $\forall j$).
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- We look for categorical crepant resolution by taking categorical minimal resolutions.
Type $A_1$ case (minimal resolution)

$X$: type $A_1$, ordinary double point, cone over $n - 1$-dimensional smooth quadric hypersurface $E \subset \mathbb{P}^n$. 
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1. If $n = 2m$,
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   $\Sigma_E$, $\Sigma_E^+$, $\Sigma_E^−$ spinor bundles. [Kapranov]
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- $\text{Perf}(X) \subset D^b(\text{Coh}(X))$: triangulated subcategory of perfect complexes (locally finite complexes of locally free sheaves).
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1. $S^D_X(c) = c[2]$, if $n = 2m$. (relatively 2-Calabi-Yau category)
2. $S^D_X(c) = c[3]$, if $n = 2m + 1$. (relatively 3-Calabi-Yau category)
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$\Rightarrow \{ a \in \mathcal{D}_X | f_*a = 0 \} = \langle c \rangle$ or $= \langle c^+, c^- \rangle$. 
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- Both are relatively 2-Calabi-Yau categories: $S_{\mathcal{D}(\mathcal{X})}(c) = c[2]$, and $S_{\mathcal{D}(\mathcal{X})}(c^\pm) = c^\pm[2]$.
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- Canonical divisors: $K_Y = f^* K_X + E_1 + E_2$.
- Corresponding SOD: there exists a triangulated subcategory (categorical minimal resolution) $\mathcal{D}_X$ s.t.
  \[ D^b(\text{Coh}(Y)) = \langle \mathcal{O}_Y(E_2)/\mathcal{O}_Y, \mathcal{O}_Y(E_1 + E_2)/\mathcal{O}_Y, \mathcal{D}_X \rangle. \]
There are sheaves $c_1, c_2, c_3$ supported on $E_1 \cup E_2$.

$$0 \rightarrow c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow 0.$$
Type $E_6$ case (Calabi-Yau property)

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- $S_{\mathcal{D}(X)}(c_1) = c_2[2], S_{\mathcal{D}(X)}(c_2) = c_3[2], S_{\mathcal{D}(X)}(c_3) = c_1[3]$.

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- Question: Let $X$ be a variety with canonical singularities.
  Then does there exist a categorical minimal resolution whose relative part has a fractionally crepant filtration?